

# On some properties of $\sigma(N)^{*†}$

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## Abstract

We show asymptotic upper and lower bounds for the greatest common divisor of  $N$  and  $\sigma(N)$ . We also show that there are infinitely many integers  $N$  with fairly large g.c.d. of  $N$  and  $\sigma(N)$ .

## 1 Introduction

We denote by  $\sigma(N)$  the sum of divisors of  $N$ .  $N$  is said to be perfect if  $\sigma(N) = 2N$  and multiperfect if  $\sigma(N) = kN$  for some integer  $k$ . It is not known whether or not an odd perfect/multiperfect number exists. There are known many results which must be satisfied by such a number. But these results are far from solving whether or not an odd perfect/multiperfect number exists.

Instead, we consider an analog of perfect/multiperfect numbers. By definition, we can easily see that  $N$  is multiperfect if and only if  $(N, \sigma(N)) = N$ . On the other hand, it is clear that  $(N, \sigma(N)) \leq N$  for any integer  $N$ . So we can suppose an integer  $N$  is close to being perfect if  $(N, \sigma(N))$  is large relatively to  $N$ .

Since  $(N, \sigma(N)) = 1$  and  $(N(N+1), \sigma(N(N+1))) \geq N \geq (N(N+1))^{1/2} - 1$  if  $N$  is prime, we find that the behavior of  $(N, \sigma(N))$  is very irregular.

However, if we suppose that  $N$  is out of some set of density zero, we can obtain some nontrivial estimates. In this direction, Erdős[1] shows that  $(N, \varphi(N)) > 1$  for almost all  $N$ . His proof can be easily modified to prove

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that  $(N, \sigma(N)) > 1$  for almost all  $N$ . Kátai and Subbarao[3] shows that for any given integer  $l$ , the inequality  $(N, \sigma(N + l)) > 1$  holds for almost all  $N$ . Before stating our result, we note that our arithmetic function does not need to be the sum-of-divisors function. Indeed, we only require that  $\sigma$  is multiplicative,  $\sigma(p) = p + 1$  for any prime  $p$ , and  $\sigma(p^e) < p^{O(e)}$  uniformly for  $p, e$  with  $p$  prime. For example, the sum of unitary divisors function  $\sigma^*(N)$  still works( $d$  is called to be an unitary divisor of  $N$  if  $d$  divides  $N$  and  $(d, N/d) = 1$ . Hence  $\sigma^*(N)$  is multiplicative and  $\sigma^*(p^e) = p^e + 1$ ).

**Theorem 1.1.** *If  $f(N)$  tends to infinity as  $N$  does, then we have*

$$(N, \sigma(N)) \leq (\log \log N)^{f(N)} \quad (1)$$

*out of a set(depending on  $f$ ) of density zero.*

**Theorem 1.2.** *There exists some constant  $c$  such that*

$$(N, \sigma(N)) \geq (\log \log \log N)^c \quad (2)$$

*holds for almost all integers  $N$ .*

In these two theorems,  $N$  runs over all integers. We can see that Theorem 1.1 remains valid if  $N$  runs over shifted primes.

**Theorem 1.3.** *If  $f(N)$  tends to infinity as  $N$  does, then we have*

$$(p + a, \sigma(p + a)) \leq (\log \log(p + a))^{f(p+a)} \quad (3)$$

*for any prime  $p$  out of a set(depending on  $f$  and  $a$ ) of density zero.*

Above three theorems are results valid for almost all members. To a contrary direction, it is interesting to find a function  $f(N)$  which grows relatively fast such that  $(N, \sigma(N)) > f(N)$  for infinitely many integers  $N$ . As mentioned before,  $f(N) = N^{1/2} - 1$  works. Using Theorem 1.3, we can prove the following result.

**Theorem 1.4.** *For any  $\epsilon > 0$ , there exist infinitely many integers  $N$  such that*

$$(N, \sigma(N)) \geq N^{2/3-\epsilon}. \quad (4)$$

## 2 Notations and Preliminary Lemmas

Many of our notations and lemmas are due to Katai and Wijsmuller [4]. We denote by  $P(n), p(n)$  the largest and smallest prime factor of  $n$  respectively.

For the variable  $x$ , let  $x_1 = \log x$ ,  $x_2 = \log x_1, \dots$ . We denote by  $c$  some positive constant not necessarily same at every occurrence. Let

$$s(x, k) = \sum_{p \leq x, p \equiv -1 \pmod{k}} \frac{1}{p}. \quad (5)$$

We cite some lemmas from [4]. Lemmas 2.1 and 2.2 are Lemma 1 in [4]. Lemma 2.3 is Lemma 2 in [4].

**Lemma 2.1.** *Let  $Y_x$  and  $E_x$  tend to infinity as  $x$  does. Write  $n = n_1 n_2$  for each integer  $n$  such that  $P(n_1) \leq Y_x < p(n_2)$ . Let*

$$S_1(x) = \{n \mid n \leq x, n_1 \geq Y_x^{E_x}\}.$$

*Then  $\#S_1(x)/x$  tends to zero as  $x$  goes to infinity.*

**Lemma 2.2.** *Let  $Y_x$  and  $E_x$  tend to infinity as  $x$  does. Write  $n = n_1 n_2$  in the same way as in Lemma 2.1. Let*

$$S_2(x) = \{n \mid n \leq x, n_2 \text{ is not square free}\}.$$

*Then  $\#S_2(x)/x$  tends to zero as  $x$  tends to infinity.*

**Lemma 2.3.** *Uniformly in  $k$  and  $x \geq e^2$ , we have*

$$s(x, k) \ll \frac{x_2}{\varphi(k)}.$$

For the purpose to prove Theorem 1.3, we need corresponding results of these lemmas concerning the set of shifted primes.

**Lemma 2.4.** *If  $E_x$ ,  $Y_x$ ,  $E_x/\log \log Y_x$ ,  $x/Y_x$  tend to infinity as  $x$  does, we have*

$$\sum_{y \geq Y_x^{E_x}, P(y) < Y_x} \pi(x, y, -a) = o(\pi(x)). \quad (6)$$

*Proof.* We divide the sum into two parts according to whether  $y$  is large or small.

For  $q < x^{1/2}$ , by the Brun-Titchmarsh theorem, we have

$$\begin{aligned} \sum_{Y_x^{E_x} \leq y < x^{1/2}, P(y) < Y_x} \pi(x, y, -a) &\ll \frac{x}{x_1} \sum_{Y_x^{E_x} \leq y < x^{1/2}, P(y) < Y_x} \frac{\log x}{\varphi(y) \log(x/y)} \\ &\ll \frac{x}{x_1} \sum_{Y_x^{E_x} \leq y < x^{1/2}, P(y) < Y_x} \frac{1}{\varphi(y)}. \end{aligned} \quad (7)$$

It is well known that the number of integers  $\leq X$  with largest prime factor  $\leq Y$  is  $\ll X \exp(-(\log X)/(\log Y))$ . It is also well known that  $\varphi(n) \gg n/\log \log n$ . So partial summation gives

$$\sum_{Y_x^{E_x} \leq y < x^{1/2}, P(y) < E_x} \frac{1}{\varphi(y)} \ll \log \log(Y_x^{E_x}) \exp(-\log(Y_x^{E_x})/\log Y_x) + \int_{Y_x^{E_x}}^{x^{1/2}} \frac{\log \log t}{t \exp(\frac{\log t}{\log Y_x})} dt. \quad (8)$$

The first term is

$$\leq \frac{\log E_x + \log \log Y_x}{e^{E_x}} \rightarrow 0 \quad (9)$$

as  $x$  tends to infinity since  $(\log \log Y_x)/E_x \rightarrow 0$  as  $x \rightarrow \infty$ .

For the same reason, we have

$$\log \log \log t / \log t \leq \log \log(E_x \log Y_x) / (E_x \log Y_x) = o(1/\log Y_x) \quad (10)$$

and therefore we can majorize  $\log \log t$  by  $c \exp(\log t/(2 \log Y_x))$ . Now the above integration is

$$\begin{aligned} &\leq \int_{Y_x^{E_x}}^{x^{1/2}} \frac{1}{t \exp(\frac{\log t}{2 \log Y_x})} dt = \int_{Y_x^{E_x}}^{x^{1/2}} t^{-(1+\frac{1}{2 \log Y_x})} dt \\ &\ll (\log Y_x) Y_x^{-E_x/(2 \log Y_x)} \\ &\ll \frac{\log Y_x}{\exp c E_x} \rightarrow 0 \end{aligned} \quad (11)$$

as  $x$  tends to infinity. Substituting estimations (9) and (11) into (8), we have

$$\sum_{Y_x^{E_x} \leq y < x^{1/2}, P(y) < Y_x} \pi(x, y, -a) = o(x/x_1) = o(\pi(x)). \quad (12)$$

For  $y \geq x^{1/2}$ , we obtain a trivial estimate

$$\sum_{y \geq x^{1/2}, P(y) < Y_x} \pi(x, y, -a) \ll x \sum_{y \geq x^{1/2}, P(y) < Y_x} \frac{1}{y}. \quad (13)$$

Using partial summation in a similar way to the case  $y < x^{1/2}$ , we have

$$\begin{aligned} \sum_{y \geq x^{1/2}, P(y) < Y_x} \frac{1}{y} &\leq \exp(-\log x/(2 \log Y_x)) + c \int_{x^{1/2}}^{\infty} \frac{dt}{t \exp(\log t/\log Y_x)} \\ &\ll \left( x^{1/(2 \log Y_x)} + \int_{x^{1/2}}^{\infty} t^{-(1+\frac{1}{\log Y_x})} dt \right) \\ &\ll \frac{\log Y_x}{x^{1/(2 \log Y_x)}} = o\left(\frac{x}{x_1^2}\right) \end{aligned} \quad (14)$$

as  $x$  tends to infinity since  $\log Y_x = o(\log x)$  by assumption. Substituting (14) into (13), we have

$$\sum_{y \geq x^{1/2}, P(y) < Y_x} \pi(x, y, -a) = o(x/x_1) = o(\pi(x)). \quad (15)$$

Combining (12) and (15) immediately gives the estimate in the lemma.  $\square$

**Lemma 2.5.** *If  $Y_x \rightarrow \infty$  as  $x \rightarrow \infty$ , then*

$$\sum_{q \geq Y_x} \pi(x, q^2, -a) = o(\pi(x)). \quad (16)$$

*Proof.* As in the proof of Lemma 2.4, we divide the sum into two parts. For  $q \leq x^{1/4}$ , by the Brun-Titchmarsh theorem, we have

$$\begin{aligned} \sum_{Y_x \leq q < x^{1/4}} \pi(x, q^2, -a) &\ll \frac{x}{x_1} \sum_{Y_x \leq q < x^{1/4}} \frac{x_1}{q(q-1) \log(x/q)} \\ &\ll \frac{x}{x_1} \sum_{Y_x \leq q < x^{1/4}} \frac{x_1}{q^2} \\ &\ll \frac{x}{Y_x x_1} = o\left(\frac{x}{x_1}\right) = o(\pi(x)) \end{aligned} \quad (17)$$

since  $Y_x$  tends to infinity together with  $x$ .

For  $q \geq x^{1/4}$ , we obtain a trivial estimate

$$\sum_{q \geq x^{1/4}} \pi(x, q^2, -a) \ll x \sum_{q \geq x^{1/4}} q^{-2} = O(x^{3/4}) = o(\pi(x)). \quad (18)$$

Combining (17) and (18) immediately gives the stated inequality.  $\square$

**Lemma 2.6.** *Let*

$$t(x, q, a) = \sum_{p \leq x, p \equiv -1 \pmod{q}} \frac{\pi(x, p, -a)}{\pi(x+a)}. \quad (19)$$

*Then we have*

$$t(x, q, a) \ll \frac{x_2}{q} \quad (20)$$

*uniformly for  $q \leq \log x$ .*

*Proof.* As in previous lemmas, we divide the sum into two parts. For  $p \leq x/e$ , by the Brun-Titchmarsh theorem, we have

$$\sum_{p \leq x/e, p \equiv -1 \pmod{q}} \pi(x, p, -a) \leq \frac{cx}{x_1} \sum_{p \leq x/e, p \equiv -1 \pmod{q}} \frac{x_1}{(p-1) \log(x/p)}. \quad (21)$$

Let  $\beta = \log p / \log x$ . Then we have  $x_1 / \log(x/p) = 1/(1 - \beta)$  and by partial summation,

$$\begin{aligned}
& \sum_{p \leq x/e, p \equiv -1 \pmod{q}} \frac{x_1}{(p-1) \log(x/p)} \\
&= \sum_{p \leq x/e, p \equiv -1 \pmod{q}} \frac{1}{(p-1)(1-\beta)} \\
&\leq \frac{\pi(x, q, -1) \log x}{x} + \int_2^{x/e} \frac{\pi(x, q, -1)}{t^2 (1 - \frac{\log t}{\log x})} dt \\
&\ll \frac{1}{q} \left( 1 + \int_2^{x/e} \frac{dt}{t \log t (1 - \frac{\log t}{\log x})} \right). \tag{22}
\end{aligned}$$

Setting  $u = \log t$ , the last integration can be computed and estimated as follows:

$$\begin{aligned}
& \int_2^{x/e} \frac{dt}{t \log t (1 - (\log t)/(\log x))} \\
&= \int_{\log 2}^{\log(x/e)} \frac{du}{u(1 - (u/\log x))} \\
&= \int_{\log 2}^{\log(x/e)} \frac{1}{u} + \frac{\log x}{1 - (u/\log x)} du \\
&= \log \frac{\log(x/e)}{\log 2} + \log(1 - \frac{\log 2}{\log x}) - \log(1 - \frac{(\log x) - 1}{\log x}) \\
&\ll \log \log x. \tag{23}
\end{aligned}$$

Thus we can bound (21) by

$$\sum_{p \leq x/e, p \equiv -1 \pmod{q}} \pi(x, p, -a) \ll \frac{xx_2}{qx_1}. \tag{24}$$

For  $p \leq x/e$ , we obtain a trivial estimate

$$\sum_{x/e \leq p \leq x, p \equiv -1 \pmod{q}} \pi(x, p, -a) \leq x \sum_{x/e \leq p \leq x, p \equiv -1 \pmod{q}} \frac{1}{p} = O\left(\frac{x}{qx_1}\right) \tag{25}$$

observing that  $\log \log x - \log \log(x/e) = 1/x_1$ . Combining (24) and (25), we obtain the stated inequality.  $\square$

### 3 Proof of Theorem 1.1

Let  $g(N)$  be an arbitrary function tending to infinity. We shall show that the number of integers  $n \leq x$  such that  $(n, \sigma(n)) > (\log \log x)^{g(x)}$

is  $o(x)$ . We can easily derive the theorem from this statement. Let  $g(x) = f(\exp \exp(\log \log x)^{1/2})/2$ . Then  $(\log \log x)^{g(x)} < (\log \log n)^{f(n)}$  if  $\exp \exp n > (\exp \exp x)^{1/2}$ . Hence the number of integers  $n \leq x$  such that  $(n, \sigma(n)) > (\log \log n)^{f(n)}$  is at most  $o(x) + (\exp \exp x)^{1/2} = o(x)$ . Hence the theorem is proved.

By Lemmas 2.1 and 2.2, we have  $\#S_1(x) = o(x)$  and  $\#S_2(x) = o(x)$ . So we may assume  $n$  belongs to none of these sets.

We set  $Y_x = E_x = x_4$ . Then  $\sigma(p^e) < x_2$  if  $p \leq Y_x$  and  $e \leq E_x$ . Let  $q \geq x_2$  be a prime dividing  $(n, \sigma(n))$ . Then there exists a prime power divisor  $p^e$  such that  $p^e \parallel n$  and  $q \mid \sigma(p^e)$ . If  $p \leq Y_x$ ,  $p^e \leq Y_x^{E_x}$  since  $n$  does not belong to  $S_1$ . Hence  $\sigma(p^e) < x_2 \leq q$ . Thus  $p$  must be greater than  $Y_x$ . Now, since  $n$  does not belong to  $S_2$ ,  $p^2$  does not divide  $n$ . Hence  $e = 1$  and we have  $p \equiv -1 \pmod{q}$ . Therefore Lemma 2.3 gives that the number of  $n \leq x$  such that  $q \mid (n, \sigma(n))$  is bounded by

$$x \sum_{\substack{p \leq x, p \equiv -1 \pmod{q}}} \frac{1}{qp} = \frac{xs(x, q)}{q} \ll x \frac{x_2}{q^2}. \quad (26)$$

Hence we find that the number of  $n \leq x$  such that  $q$  divides  $(n, \sigma(n))$  for some  $q \geq x_2$  is at most

$$cx \sum_{q \geq x_2} \frac{x_2}{q^2} \ll \frac{xx_2}{x_2 \log x_2} = O\left(\frac{x}{x_3}\right). \quad (27)$$

It follows that  $(n, \sigma(n))$  divides  $n_3$  with at most  $o(x)$  exceptions, where  $n = n_3 n_4$  with  $P(n_3) < x_2 < p(n_4)$ . Our statement follows by observing that the number of integers  $n \leq x$  with  $n_3 \geq x_2^{g(x)}$  is at most  $o(x)$  by Lemma 2.1. This completes the proof of Theorem 1.1.

## 4 Proof of Theorem 1.2

Denote by  $N(x, k, Q)$  the number of integers  $n \leq x$  such that  $n$  is divisible by at most  $k$  primes  $\leq Q$ . Brun's sieve gives  $N(x, 0, Q) = O(x/\log Q)$  uniformly for  $Q, x$  with  $Q < x$  and a simple inductive argument immediately gives

$$N(x, k, Q) = O(x(c + \log \log Q)^k / \log Q) \quad (28)$$

uniformly for  $Q, x, k$  with  $Q^{k+1} < x$ .

Let  $Q$  and  $l$  be integers satisfying  $x > Q^{l+1}$ .

Let  $q_1 < q_2 < \cdots < q_l < Q$  be distinct primes and  $M(y, q_1, \dots, q_l)$  denote the number of integers  $n < y$  such that  $\sigma(n)$  is divisible by none of  $q_1, \dots, q_l$ . If  $p$  is a prime  $\equiv -1 \pmod{q}$ , then  $p$  does not divide  $n$  or  $p^2$  divides  $n$ . Hence it follows from Brun's sieve and the Prime Number Theorem in arithmetic progressions in the form of Theorem 9.6 in Karatsuba [2] that

$$M(y, q_1, \dots, q_l) = O\left(\frac{ly}{(\log y)^{1/(Q-1)}}\right) \quad (29)$$

uniformly for  $Q, y$  with  $Q < 2 \log y$ .

Let  $q_1 < q_2 < \cdots < q_l < Q$  be distinct primes. Let  $R_{xj}$  be functions of  $x$  such that  $R_{xj} < x^{1/2l}$ . Then the number of integers  $n$  such that  $n < x$ ,  $q_1 \cdots q_l \mid n$  and  $q_1 \cdots q_l \nmid \sigma(n)$  is at most

$$\begin{aligned} & \sum_{e_1, \dots, e_l \geq 1} M\left(\frac{x}{q_1^{e_1} \cdots q_l^{e_l}}, q_1, \dots, q_l\right) \\ & \leq \sum_{e_1, \dots, e_l \geq 1, \forall j, q_j^{e_j} \leq R_{xj}} M\left(\frac{x}{q_1^{e_1} \cdots q_l^{e_l}}, q_1, \dots, q_l\right) \\ & \quad + \sum_{j=1}^l \sum_{e_1, \dots, e_l \geq 1, q_j^{e_j} \geq R_{xj}} \frac{x}{q_1^{e_1} \cdots q_l^{e_l}} \\ & = \Sigma_1 + \Sigma_2, \text{ say.} \end{aligned} \quad (30)$$

By (29), we see that  $\Sigma_1$  is

$$O\left(\frac{lx}{(q_1 - 1) \cdots (q_l - 1)x^{1/(Q-1)}}\right) \quad (31)$$

since  $Q < 2 \log(x^{1/2}) < \log(x/(R_{x1} \cdots R_{xl})) < \log(x/(q_1^{e_1} \cdots q_l^{e_l}))$ . A trivial argument gives

$$\Sigma_2 \leq \sum_{j=1}^l \frac{xq_j}{R_{xj}(q_1 - 1) \cdots (q_l - 1)} \ll lx^{1-1/2l}. \quad (32)$$



Combining these estimates, we obtain

$$\sum_{e_1, \dots, e_l \geq 1} M\left(\frac{x}{q_1^{e_1} \cdots q_l^{e_l}}, q_1, \dots, q_l\right) \ll \left(\frac{lx}{(q_1 - 1) \cdots (q_l - 1)x_1^{1/(Q-1)}} + lx^{1-1/(2l)}\right). \quad (33)$$

Hence the number of integers  $n < x$  such that there exist distinct primes  $q_1 < q_2 < \cdots < q_l < Q$  satisfying  $q_1 \cdots q_l \mid n$  and  $q_1 \cdots q_l \nmid \sigma(n)$  is bounded by

$$\sum_{q_1 < q_2 < \cdots < q_l < Q} c \left(\frac{lx}{(q_1 - 1) \cdots (q_l - 1)x_1^{1/(Q-1)}} + lx^{1-1/(2l)}\right) \ll \frac{lx(\log \log Q + c)^l}{x_1^{1/(Q-1)}} + \frac{lQ^l x}{x^{1/2l}}. \quad (34)$$

We observe that  $Q = x_2/x_3$  and  $l = cx_4/x_5$  with  $c > 0$  sufficiently small satisfy our conditions and the right-hand side of (34) is  $o(x)$ . By (28), the number of integers  $n < x$  divisible by at most  $l - 1$  distinct primes smaller than  $Q$  is  $o(x)$ . Now the remaining integers  $n$  have the property that  $(n, \sigma(n))$  has at least  $l$  distinct prime factors. Hence  $(n, \sigma(n)) > l^c > x_3^c$ . This completes the proof.

## 5 Proof of Theorem 1.3 and 1.4

Set  $Y_x = E_x = x_4$  as in the proof of Theorem 1.1. We easily see that this choice satisfies the condition of Lemma 2.4. Proceeding in the same way as our proof of Theorem 1.1, we immediately obtain Theorem 1.3 using Lemmas 2.4-2.6 instead of Lemmas 2.1-2.3.

Let  $N = p(p+1)m$  where  $m$  is the largest divisor of  $\sigma(p+1)$  relatively prime to  $p+1$ . The proof of Theorem 1.3 shows that  $P((p+1, \sigma(p+1))) \leq n_1$  and  $n_1 < (\log \log(p+1))^{f(p+1)}$  for almost all prime  $p$ , where  $p+1$  is decomposed into  $n_1 n_2$  such that  $P(n_1) \leq \log \log(p+1) < p(n_2)$ . Hence if  $\epsilon$  is an arbitrary positive real number,  $(p+1, \sigma(p+1)) = o(x^\epsilon)$  holds for almost all prime  $p$ .

Hence there exists infinitely many prime  $p$  such that  $m > p^{1-\epsilon}$ . If we choose such  $p$ , then  $(p+1)m$  divides  $\sigma(N)$ . Hence  $(p+1)m$  divides  $(N, \sigma(N))$  and clearly  $(p+1)m > N^{2/3-\epsilon}$ . This completes the proof.

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